SIO 214a Introduction to Fluids Problem Set 4
Due Monday, 22 October 2007

1. Consider the flow of mud down a plane that is inclined by an angle θ from the horizontal. First we solve the simpler problem of the flow of fluid down the plane, then we modify the stress-rate of strain relationship for mud. Use a coordinate system where x is the coordinate parallel to the plane, z increases away from the surface of the plate, where it is zero. Assume that the flow is two-dimensional, so there are no changes in y. Further assume that the plane is very long, so that nothing changes in x, that is $\partial/\partial x = 0$. The layer will have constant thickness H. The viscosity of the air above $z = H$ is taken to be zero. Finally, assume that the flow is steady. Start with the continuity and momentum equations for a deformable medium:
$$\frac{\partial u_i}{\partial x_i} = 0 \tag{1}$$
$$\frac{Du_i}{\partial t} = \rho g_i + \frac{\partial \tau_{ij}}{\partial x_j} \tag{2}$$
Remember that in the indicial notation, a repeated index implies summation, so that the first equation is shorthand for
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3}$$
g_i is the component of gravity in the ith direction

   - With the assumptions given above, show that the x component of the momentum equation reduces to:
$$\frac{\partial \tau}{\partial z} = -\rho g \sin\theta \tag{4}$$
where τ will be used to denote the only non-zero component of shear stress, τ_{xz}. Integrate this equation with a boundary condition at $z = H$ to find $\tau(z)$.
   - If the flowing medium is fluid, $\tau = \mu \partial u/\partial z$. Write appropriate boundary conditions on u at $z = 0$, and find $u(z)$.
   - An idealized model of the mud is that if τ is less than a critical value τ_c, it remains solid ($\partial u/\partial z = 0$), otherwise $\tau = \tau_c + \mu \partial u/\partial z$. This means that the shear stress has to exceed the constant value τ_c before the mud starts to flow. The momentum equation 4 remains unchanged. In this case, show that the angle the plane makes with the horizontal has to be at least $\theta_c = \arcsin\left[\tau_c/(\rho g H)\right]$ before the mud flows.
   - If $\theta > \theta_c$, the solution can be found in two different regions, the region extending from $z = 0$ to $z = h < H$, where $\tau > \tau_c$, and the region $h \le z \le H$ where $\tau \le \tau_c$. Find h/H in terms of θ and θ_c. Use equation 4 to get an expression for u is the lower layer, then evaluate the (constant) velocity $u = U$ in the upper layer. To find u in the lower layer, you will need to come up with a boundary condition at $z = h$.

2. A steady two-dimensional viscous flow with viscosity μ is given by the streamfunction
$$\psi = Axy$$
where $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$.

- Sketch the streamlines (for $A > 0$) in the x, y plane. Include arrows to show the direction of the flow on each streamline.
- Neglecting gravity, the momentum equation is:

$$\frac{Du_i}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\mu}{\rho}\nabla^2 u_i$$

  where $u_i = u, v$ and $x_i = x, y$. Integrate this equation to obtain a relationship between pressure, A, x and y. Assume that $p = p_0$ at $x = y = 0$.
- If the line $x = 0$ represents a solid wall, could the streamfunction represent the flow near $x = y = 0$? Explain briefly.

3. Consider the x and y-components of the momentum equation for the two-dimensional flow of an incompressible, inviscid fluid. Combine the equations in such a way as to eliminate the pressure term and, using the conservation of mass equation, show that

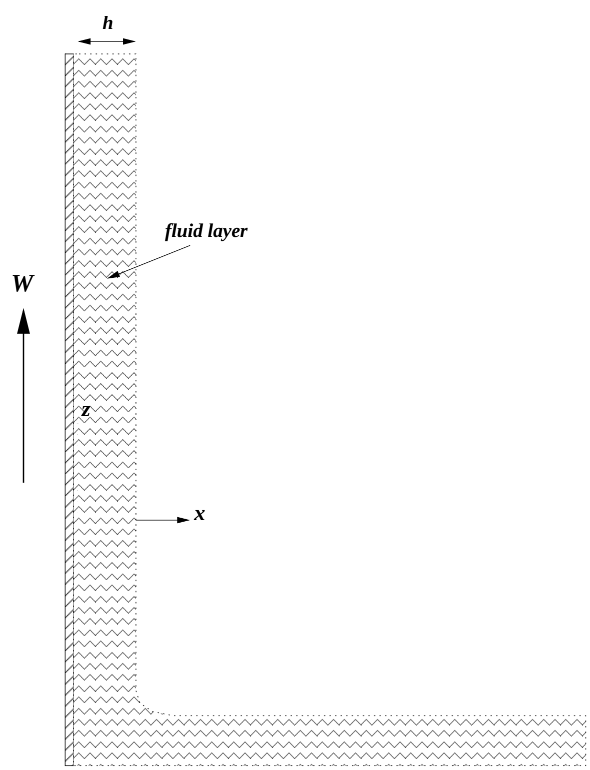
$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0$$

where ω represents the vorticity

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

What can you conclude about the vorticity of a fluid element in such a flow?

4. A wide moving belt rises out of a container, drawing fluid of constant density ρ and viscosity μ with it. The flow is steady and the belt moves upward with velocity W. Viscous forces result in the belt



carrying up a thin film of fluid, of thickness h. You may assume that the belt is quite long, so that variables are independent of z, and that the pressure just outside the fluid layer is constant. The shear stress in the fluid at the air-fluid interface is zero.

- Show that the z-component of the momentum equation reduces to

$$0 = -\rho g + \mu \frac{\partial^2 w}{\partial x^2}$$

  what are the other two components? what are the boundary conditions on w?
- Find the velocity w as a function of z.
- Find the average film velocity.